ON THE APPROXIMATE REPRESENTATION OF AN INDEFINITE INTEGRAL AND THE DEGREE OF CONVERGENCE OF RELATED FOURIER'S SERIES*

BY

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It has been proved by the author† that if $f(x)$ is a function of period 2π which satisfies a Lipschitz condition, it is possible to construct for each positive integral value of n a trigonometric sum $T_n(x)$, of order n or less, such that $|f(x) - T_n(x)|$ remains always inferior to a constant multiple of $1/n$. It will be convenient to use the notation $\mathfrak{T}_n(x)$ as an abbreviation for the words "a trigonometric sum in x of the nth order at most." The symbol $\mathfrak{T}_n$ is accordingly not to be understood as a functional operator. It satisfies, for example, the relations $\mathfrak{T}_n(x) + \mathfrak{T}_n(x) = \mathfrak{T}_n(x)$, $\mathfrak{T}_n(x) + \mathfrak{T}_{n+1}(x) = \mathfrak{T}_{n+1}(x)$. With this convention, we may express the theorem stated above by the equation

$$f(x) = \mathfrak{T}_n(x) + O\left(\frac{1}{n}\right),$$

where $O(1/n)$ stands for a function which never exceeds a constant multiple of $1/n$.

It will be shown below, as the fundamental result of the first part of the paper, that the theorem quoted has the following consequence: If $f(x)$ is expressible in the form

$$f(x) = \mathfrak{T}_n(x) + O(\varphi(n)),$$

then there exists for any indefinite integral of $f(x)$ a representation of the form

$$\int f(x)\,dx = \mathfrak{T}_n(x) + O\left(\frac{\varphi(n)}{n}\right),$$

provided that the obviously necessary condition is satisfied, that $f(x)$ be such as to make the integral periodic.

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† *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, Dissertation, Göttingen, 1911; pp. 42–46. *On approximation by trigonometric sums and polynomials*, these Transactions, vol. 13 (1912), pp. 491–515. The former paper will be referred to as *Thesis*, the latter as A.

The existence of such a relation is at once suggested by the fact, established by the author* by a modification of the method used to obtain the first result, that if $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition, then

$$f(x) = \mathfrak{T}_n(x) + O\left(\frac{1}{n^k}\right).$$

This generalized proposition is now seen to be a consequence of the simpler one. But we shall obtain more than a new proof of the generalized theorem, namely, a convenient means, not previously recognized, of determining for all values of k numerical limits for the constant multipliers that are involved implicitly in the notation $O(1/n^k)$. Only the cases $k = 1$ and $k = 2$ had been subjected to detailed investigation in this regard, and the result obtained in the latter of these cases will now be considerably improved, with the aid of a modification of the main theorem.

As a further application of the method a result of still greater generality will be deduced.

The considerations thus far outlined form the substance of the first part of the paper, dealing with approximation by means of trigonometric sums of the nth order. In the second part, the problem of approximation by polynomials of the nth degree will be discussed in the same manner, and corresponding results will be obtained, though with some divergence in the details.

The problems discussed in the third division of the paper, relating mainly to Fourier's series, are at the outset of a somewhat different nature. Their connection with what precedes will be pointed out at the beginning of that section, just before they are taken up in detail.

I. Trigonometric Approximation.

The notation $\mathfrak{T}_n(x)$ has already been explained. We shall introduce also the symbol $\theta(n, x)$ (or $\theta(x)$) to represent "a function of n and x (or of x alone) which never exceeds 1 in absolute value," not as a functional symbol, but as a shorthand expression to be used without alteration to refer to as many different functions of the character indicated as may occur. Every equation in which this symbol appears, in the present section, is understood to hold for all real values of x. With this explanation, we can state the theorem which forms our starting-point as follows:

Theorem I: *There exists an absolute constant K_1 having the following property: If $f(x)$ is a function of the real variable x, of period 2π, which everywhere satisfies the Lipschitz condition*

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

* *Thesis,* Satz VII; A, Theorem III. The proof in A is simpler than that in the thesis.

then, for every positive integral value of n,

$$f(x) = \mathfrak{T}_n(x) + \frac{K_1\lambda}{n}\theta(n, x).$$

The meaning of the last line is that the function $f(x)$ can be represented by a trigonometric sum of the nth order or lower with an error not exceeding $K_1\lambda / n$. For the proof of this theorem, the reader is referred to my thesis, pp. 42–46, and the paper A, Theorem I. We assume also the following theorem from A (Theorem VI):

The constant K_1 may be given the numerical value 3, and this is not the smallest admissible value; but a value smaller than $\pi / 2$ would be incorrect.

The fundamental result to be established in this section is

THEOREM II: *If $f(x)$ is an integrable* function of period 2π, the integral of which over an interval of length 2π is zero, so that an indefinite integral of $f(x)$ is periodic, and if*

$$f(x) = \mathfrak{T}_n(x) + \epsilon\theta(x),$$

ϵ being a constant, then

$$\int f(x)\,dx = \mathfrak{T}_n(x) + \frac{2K_1\epsilon}{n}\theta(x).$$

The constant K_1 is that of Theorem I. The formulation above has been chosen to emphasize the fact that the theorem may be stated and proved without reference to more than a single value of n. Of course it follows at once that if $f(x) = \mathfrak{T}_n(x) + \varphi(n)\theta(n, x)$, for several or infinitely many or all values of n, then

$$\int f(x)\,dx = \mathfrak{T}_n'(x) + \frac{2K_1\varphi(n)}{n}\theta(n, x),$$

for these values of n.

For the proof of the theorem, set

$$f(x) = T_n(x) + R_n(x),$$

where $T_n(x)$ is a particular trigonometric sum of the nth order at most, and

$$|R_n(x)| \overline{\gtrless} \epsilon.$$

Let a_0 be the constant term of $T_n(x)$. There is no reason why this term should necessarily be zero, as would be the case with the constant term in the Fourier's development of $f(x)$. However, the magnitude of a_0 is not unrestricted; we can be certain that $|a_0| \overline{\gtrless} \epsilon$. For if we write down the relation

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi}[T_n(x) - a_0]\,dx + \int_0^{2\pi}[a_0 + R_n(x)]\,dx,$$

* In the ordinary sense or in the sense of LEBESGUE.

the integral of $f(x)$ is zero by hypothesis, and that of $T_n(x) - a_0$ is zero because the integrand is a trigonometric sum without constant term, and consequently

$$\int_0^{2\pi} [a_0 + R_n(x)]\,dx = 0,$$

from which it appears that $|a_0|$ can not exceed the maximum of $|R_n(x)|$.

It is clear that if the theorem is true for one indefinite integral of $f(x)$ it is true for all, since two indefinite integrals of the same function differ only by a trigonometric sum of order zero. Let us consider the integral

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x [T_n(x) - a_0]\,dx + \int_0^x [a_0 + R_n(x)]\,dx.$$

The first of these two terms is a trigonometric sum of the same order as $T_n(x)$. The second is a continuous function of x satisfying a Lipschitz condition with coefficient 2ϵ. For

$$\left|\int_0^{x_2} [a_0 + R_n(x)]\,dx - \int_0^{x_1} [a_0 + R_n(x)]\,dx\right| = \left|\int_{x_1}^{x_2} [a_0 + R_n(x)]\,dx\right|$$

$$\overline{\gtrless} \left|\int_{x_1}^{x_2} (\epsilon + \epsilon)\,dx\right| = 2\epsilon\,|x_2 - x_1|.$$

Hence the term has the form $\mathfrak{T}_n(x) + \theta(x) \cdot 2K_1\epsilon/n$, by Theorem I, and $F(x)$ has the same form, as was asserted in the theorem to be proved.

It may be observed that if it had been known through any specialization of the problem that a_0 was zero, the factor 2 would not have entered, and we should have had an error not exceeding $K_1\epsilon/n$.

If we suppose that $f(x)$ has a $(k-1)$th derivative satisfying the Lipschitz condition

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \overline{\gtrless} \lambda\,|x_2 - x_1|,$$

we may apply Theorem I to the representation of $f^{(k-1)}(x)$ by a trigonometric sum, and then Theorem II to the representation of $f^{(k-2)}(x), \cdots, f'(x)$, $f(x)$, successively,* and deduce the theorem that $f(x)$ has the form $\mathfrak{T}_n(x) + \theta(n, x) \cdot 2^{k-1} K_1^k \lambda / n^k$. Thus Theorem III of the article A, which differs from this in having a new undetermined constant K_k for each value of k instead of the expression $2^{k-1} K_1^k$, is proved more simply than

* Of course the condition that the integral over an interval of length 2π shall be zero is not imposed on $f(x)$ here, but concerns only the derivatives, by which it is satisfied automatically.

before, and at the same time rendered considerably more definite. If we give K_1 the value 3, which we know to be admissible, we have at once numerical values for all the constants K_k.

It is not difficult, however, to improve materially upon the values obtained in this way, namely, to justify the omission of the factor 2^{k-1} and the substitution of 3^k for K_k. Up to the present we have used Theorem I as it was stated, without reference to the method by which it was proved. If it should be shown by any method whatever that Theorem I remains true with a smaller value of K_1, then this value could be substituted for K_1 throughout the preceding work, and everything that has been said would still be true. We have not derived any benefit from the fact that Theorem I was proved (in the article A, as also in the thesis) by constructing trigonometric sums of a special form which furnished the degree of approximation desired. That fact can however be made very useful.

It has been remarked that if the given approximating function in the hypothesis of Theorem II lacks the constant term, the factor 2 in the conclusion can be dispensed with. The trigonometric sums used to prove Theorem I were of the form

$$C\int_{-\pi}^{\pi} f(v)\left[\frac{\sin\frac{1}{2}m(v-x)}{m\sin\frac{1}{2}(v-x)}\right]^4 dv,$$

where C is a constant and m is a positive integer. It was shown that the fourth power in the integrand is a trigonometric sum in $v - x$, so that, on expanding the several terms and integrating, a trigonometric sum in x is obtained. And now it is clear that if the integral of $f(x)$ from $-\pi$ to π is zero, the constant term in the expression above, regarded as a trigonometric sum in x, will vanish.

In fact, any terms will be lacking which may be lacking in the Fourier's expansion* of $f(x)$. For the fourth power involved, being an even function of $v - x$, and being at the same time a trigonometric sum in this argument, is merely a sum of cosines of multiples of $v - x$. The terms in $\cos px$ and $\sin px$ in the sum obtained on integrating are of the forms

$$C\int_{-\pi}^{\pi} f(v)\cos pv\cos px\,dv, \qquad C\int_{-\pi}^{\pi} f(v)\sin pv\sin px\,dv,$$

and will vanish if the corresponding Fourier's coefficients do so. In fact, it will be seen upon closer inspection that our method amounts to a particular kind of "summation" of the Fourier's series, as has frequently been pointed out in connection with similar methods before; that is, our trigonometric

* In this connection, cf. Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, second series, vol. 7 (1906), pp. 81–152 (also published separately), pp. 102–103, and the first footnote on the latter page.

sums may be regarded as obtained from the partial sums of corresponding order in the Fourier's series for $f(x)$ by multiplying the terms of the latter by constant factors, independent of the function developed, which approach unity, in the case of any particular term, as the number of terms is increased. We are in a position to state that *if $f(x)$ satisfies a Lipschitz condition with coefficient λ, then $f(x)$ can be represented, with an error not exceeding $3\lambda / n$, by a trigonometric sum of the nth order at most, containing no terms which do not occur in the Fourier's development of $f(x)$.* Of course nothing that has been said would justify the conclusion that a smaller value of K_1 in Theorem I, which might be obtained by a different method, would be applicable here. On the other hand, any value obtained by a further refinement of the same method would be admissible.

We are concerned here only with the vanishing of the constant term. Let it be assumed that $f(x)$ is periodic, with period 2π, and has a derivative of order $(k-1)$ which satisfies a Lipschitz condition with coefficient λ. Then $f^{(k-1)}(x) = \mathfrak{T}_n(x) + \theta(n, x) \cdot 3\lambda / n$. If $k = 1$, nothing more is desired. If $k > 1$, then $f^{(k-1)}(x)$ has an integral which is periodic, and the constant term in the sum $\mathfrak{T}_n(x)$ above may be assumed to be zero. Consequently $f^{(k-2)}(x) = \mathfrak{T}_n x) + \theta(n, x) \cdot 3^2 \lambda / n^2$, by the version of Theorem II which is applicable in such cases. If $k = 2$, we stop at this point; we have $K_2 = 9$, instead of the much cruder value 20 (or 18.2) of the earlier paper A (Theorem VIII, with the text immediately preceding).

If $k > 2$, our next step is to show that in the trigonometric sum representing $f^{(k-2)}(x)$ the constant term may again be assumed to vanish. To this end, we go back to the representation of $f^{(k-1)}(x)$. We may set

$$f^{(k-1)}(x) = t_n(x) + r_n(x),$$

where $t_n(x)$ is a trigonometric sum of order n or less with no constant term, and $|r_n(x)| \overline{\overline{<}} 3\lambda / n$. By integrating, we get

$$f^{(k-2)}(x) = C + \int_0^x t_n(x)\, dx + \int_0^x r_n(x)\, dx,$$

where C is a constant. The first integral on the right is a trigonometric sum, since $t_n(x)$ lacks the constant term. The second integral is a continuous function of x satisfying a Lipschitz condition with coefficient $3\lambda / n$, and will be represented by a trigonometric sum with a corresponding degree of approximation, as was done in the proof of Theorem II; but we are not quite ready for this step. Let us set

$$C + \int_0^x t_n(x)\, dx = T_n(x) = \mathfrak{T}_n(x).$$

Then

$$f^{(k-2)}(x) = T_n(x) + \int_0^x r_n(t)\,dt.$$

Denoting the constant term in $T_n(x)$ by c_0, and subtracting and adding this term, we have

$$f^{(k-2)}(x) = T_n(x) - c_0 + c_0 + \int_0^x r_n(t)\,dt.$$

Since $T_n(x) - c_0$ is a trigonometric sum without constant term,

$$\int_0^{2\pi} [T_n(x) - c_0]\,dx = 0.$$

Furthermore,

$$\int_0^{2\pi} f^{(k-2)}(x)\,dx = f^{(k-3)}(2\pi) - f^{(k-3)}(0);$$

and this is zero, since we suppose that $k \geqq 3$, and hence $f^{(k-3)}(x)$ is really a derivative of $f(x)$ or else $f(x)$ itself, and so periodic. It follows that

$$\int_0^{2\pi} \left\{ c_0 + \int_0^x r_n(t)\,dt \right\} dx = 0.$$

This is the essential point of the present argument. It shows that the expression

$$c_0 + \int_0^x r_n(t)\,dt,$$

which satisfies the Lipschitz condition with coefficient $3\lambda / n$, can be represented by a trigonometric sum of order n at most, *with no constant term,* so that the error shall be not greater than $3^2\lambda / n^2$. Consequently $f^{(k-2)}(x)$ has a representation of the same sort.

Thus we are enabled to apply Theorem II once more in its specialized form, obtaining $f^{(k-3)}(x) = \mathfrak{T}_n(x) + \theta(n, x) \cdot 3^3 \lambda / n^3$. The reasoning of the preceding paragraph, *mutatis mutandis,* may be repeated step by step to show that here also the constant term in the trigonometric sum may be assumed to vanish, if $k \geqq 4$. And so we proceed until we get a representation of $f'(x)$ lacking the constant term, and a representation of $f(x)$, in which the presence of a constant term is immaterial, the error in the last representation being not greater than $3^k \lambda / n^k$. We have thus the remarkably simple result, which we will state at length, without abbreviation:

THEOREM III: *If $f(x)$ is a function of period 2π possessing a $(k-1)$th derivative which everywhere satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda |x_2 - x_1|,$$

then there exists for each positive integral value of n a trigonometric sum $T_n(x)$, *of the nth order at most, such that for all values of x*

$$|f(x) - T_n(x)| \leqq \frac{3^k \lambda}{n^k}.$$

The bearing of this result on the theory of the convergence of Fourier's series is indicated in Theorem V of the article A. We shall do better, however, to use another method (Theorem X below).

To prepare the way for a more general theorem, suppose that $f(x)$ is any continuous function of x with the period 2π. Corresponding to this function $f(x)$, let a function $\omega(\delta)$ of the positive variable δ be defined, so that, however x_1 and x_2 may be chosen subject to the condition $|x_2 - x_1| \leqq \delta$, we shall always have $|f(x_2) - f(x_1)| \leqq \omega(\delta)$. For our present purpose no further restriction is laid on the function ω. It follows from its definition that $\omega(\delta)$ never decreases when δ increases. The author has shown* that under these circumstances

$$f(x) = \mathfrak{T}_n(x) + \left(\frac{K_1}{2\pi} + 2\right)\omega\left(\frac{2\pi}{n}\right)\theta(n, x),$$

where K_1 is the constant of Theorem I and the relation holds for all positive integral values of n. The proof† is so simple that it may be reproduced here. We first define a function $f_1(x)$ equal to $f(x)$ at the points $x = 2s\pi / n$ for all integral values of s, and linear in each of the intervals of length $2\pi / n$ between two successive points of this set. Then,‡ on the one hand,

$$|f(x) - f_1(x)| \leqq 2\omega\left(\frac{2\pi}{n}\right)$$

for all values of x; and, on the other hand, $f_1(x)$ satisfies the conditions for the application of Theorem I, with

$$\lambda = \omega\left(\frac{2\pi}{n}\right)\Big/\frac{2\pi}{n}.$$

Hence the proposition follows at once.

By starting from this theorem, and applying Theorem II repeated a sufficient number of times, as we have done before, we deduce the following general theorem:

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* *Thesis*, p. 48.

† It is readily seen that nothing need be changed if $f(x)$ is supposed to be discontinuous, but to remain finite.

‡ The central idea of this proof is taken from LEBESGUE, who uses it in a different connection; Bulletin de la Société Mathématique de France, vol. 38 (1910), pp. 184–210; p. 202 of the volume, p. 19 of the article.

THEOREM IV: *If $f(x)$ has the period 2π and possesses a continuous kth derivative satisfying the condition that*

$$|f^{(k)}(x_2) - f^{(k)}(x_1)| \overline{\gtrless} \omega(\delta)$$

whenever $|x_2 - x_1| \overline{\gtrless} \delta$, *then*

$$f(x) = \mathfrak{T}_n(x) + O\left(\frac{1}{n^k}\omega\left(\frac{2\pi}{n}\right)\right),$$

uniformly for all values of x.

By saying that this relation holds "uniformly" we mean, naturally, that the constant multipliers implied in the O-notation are independent of x. Of course numerical values could be calculated for them without difficulty.

The result just obtained can be applied to the theory of Fourier's series in the same manner as Theorem III above.*

II. POLYNOMIAL APPROXIMATION.

Corresponding to the notation $\mathfrak{T}_n(x)$ of the first part of the paper, we shall introduce the symbol $\mathfrak{P}_n(x)$ to replace the words "a polynomial in x of the nth degree at most." The signs $\theta(x)$, $\theta(n, x)$, will be used as before, except that relations involving them are understood to hold, no longer for all real values of x, but for all values of x in the closed interval under consideration.

We set out from the following theorem, which we shall quote without proof:

THEOREM V: *There exists an absolute constant L_1 having the following property:*† *If $f(x)$ is a function which satisfies the Lipschitz condition*

$$|f(x_2) - f(x_1)| \overline{\gtrless} \lambda |x_2 - x_1|$$

throughout the closed interval $a \overline{\gtrless} x \overline{\gtrless} b$ of length l, then, for every positive integral value of n,

$$f(x) = \mathfrak{P}_n(x) + \frac{L_1 l \lambda}{n}\theta(n, x),$$

throughout the interval. Furthermore,‡ *the constant L_1 may be given the value $\frac{1}{2}K_1$, if K_1 is any number which is admissible in Theorem I; so that, in particular,*§ *L_1 may have the value $\frac{3}{2}$; but it can not have a value smaller than $\frac{1}{2}$.*

From this theorem we obtain readily

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* The theorem deduced in this way is given, for a large class of cases, on p. 92 of a recent essay of BERNSTEIN to which more extended reference will be made in a later footnote.

† *Thesis*, p. 39; A, Theorem II.

‡ A, Theorem II.

§ A, Theorem VII. There is a misprint in the statement of the theorem, and also four lines above; the symbol L_2 should be L_1.

THEOREM VI: *If, in the interval* $a \leqq x \leqq b$ *of length* l, *the function* $f(x)$ *is integrable,** *and*

$$f(x) = \mathfrak{P}_n(x) + \epsilon\,\theta(x),$$

ϵ *being a constant, then*

$$\int f(x)\,dx = \mathfrak{P}_{n+1}(x) + \frac{L_1 l \epsilon}{n+1}\theta(x).$$

According to the hypothesis concerning $f(x)$, we may write

$$f(x) = \Pi_n(x) + R_n(x),$$

where $\Pi_n(x)$ is a polynomial of the nth degree at most, and

$$|R_n(x)| \leqq \epsilon.$$

Consider the integral

$$F(x) = \int_a^x f(x)\,dx = \int_a^x \Pi_n(x)\,dx + \int_a^x R_n(x)\,dx.$$

The first term here is a polynomial of degree $n + 1$ at most. The second is a function satisfying the condition

$$\left|\int_a^{x_2} R_n(x)\,dx - \int_a^{x_1} R_n(x)\,dx\right| \leqq \epsilon\,|x_2 - x_1|;$$

so that by Theorem V, with n replaced by $n+1$, it has the form $\mathfrak{P}_{n+1}(x) + \theta(x)\cdot L_1 l\epsilon/(n+1)$. It follows that $F(x)$ has the same form.

Then a moment's reflection yields the theorem:

THEOREM VII: *If* $f(x)$ *is a function possessing a* $(k-1)$*th derivative which satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda\,|x_2 - x_1|$$

throughout the interval $a \leqq x \leqq b$ *of length* l, *then, for each integral value of* $n \geqq k$,

$$f(x) = \mathfrak{P}_n(x) + \frac{L_1^k l^k \lambda}{n(n-1)\cdots(n-k+1)}\theta(n, x).$$

First, we express $f^{(k-1)}(x)$ in the form

$$f^{(k-1)}(x) = \mathfrak{P}_{n-k+1}(x) + \theta(x)\cdot L_1 l\lambda/(n-k+1),$$

by Theorem V; and then we apply $k-1$ times in succession the theorem just obtained, to arrive at the conclusion stated.

This recalls at once Theorem IV of the paper A, which attaches to the same hypothesis the conclusion that, for $n \geqq k-1$, the function $f(x)$ has the

* In the ordinary sense or in the sense of LEBESGUE.

form $f(x) = \mathfrak{P}_n(x) + \theta(n, x) \cdot L_k l^k \lambda / n^k$, where L_k is a constant dependent only on k. If we leave out of account for the moment the fact that one of these results is stated for all values of n from $n = k - 1$, the other only for values from $n = k$ on, we see that we have a new and simple proof of the old theorem, since

$$\frac{L_1^k l^k \lambda}{n(n-1)\cdots(n-k+1)} = \frac{n^k}{n(n-1)\cdots(n-k+1)} \cdot \frac{L_1^k l^k \lambda}{n^k} \overline{\gtrless} \frac{k^k}{k!} \cdot \frac{L_1^k l^k \lambda}{n^k} \qquad (n \geqq k).$$

Further, we have a means of computing numerical values for the constants L_k which are correct if we suppose that we are concerned only with values of $n \overline{\gtrless} k$. When numerical results are desired, however, it is manifestly better to use the formally less simple statement of our present Theorem VII as it stands.

As for the representation of $f(x)$ by a polynomial of degree $k - 1$, it is obvious that $f^{(k-1)}(x)$ can be represented by a constant, namely, by its value at the middle point of the interval, with an error not greater than $\frac{1}{2} l\lambda$; and it follows from this, by repeated application of Theorem VI, that $f(x)$ admits the representation $f(x) = \mathfrak{P}_{k-1}(x) + \theta(x) \cdot \frac{1}{2} L_1^{k-1} l^k \lambda / (k-1)!$. As it is certain* that L_1 cannot be less than $\frac{1}{2}$, we may reason further that

$$\frac{L_1^{k-1} l^k \lambda}{2(k-1)!} \overline{\gtrless} \frac{L_1^k l^k \lambda}{(k-1)!} = \frac{(k-1)^k}{(k-1)!} \cdot \frac{L_1^k l^k \lambda}{(k-1)^k},$$

which is an expression of precisely the form demanded by the old theorem. If we multiply L_1^k by the larger of the quantities $k^k / k!$ and $(k-1)^k / (k-1)!$, we have a value of L_k which is applicable in that theorem without exception. For $k = 2$, the value $L_1 = \frac{3}{2}$ gives $L_2 = 4\frac{1}{2}$, which is better than the value somewhat less than 10 that was previously known.† It would be easy by means of special considerations to reduce this value still further, but we shall not undertake to do so here.

The present method throws a new light on the difference between Theorems III and IV of A, one holding for $n \overline{\gtrless} 1$ and the other only for $n \overline{\gtrless} k - 1$. This difference appears now as a consequence of the fact that the integral of a trigonometric sum (without constant term) is a trigonometric sum of the same order, while the integral of a polynomial is a polynomial of higher degree.

Lastly, we seek an analogon of Theorem IV for the case of polynomial approximation. It is known‡ that if $|f(x_2) - f(x_1)| \overline{\gtrless} \omega(\delta)$ whenever $|x_2 - x_1| \overline{\gtrless} \delta$ and x_1, x_2 are in the interval $a \overline{\gtrless} x \overline{\gtrless} b$ of length l, then $f(x) = \mathfrak{P}_n(x) + \theta(x) \cdot (L_1 + 2)\,\omega(l/n)$ for all positive integral values

* Cf. the last clause of Theorem V above.

† A, Theorem IX.

‡ *Thesis*, pp. 40–41.

of n. In this case, the interval from a to b is divided into n equal parts, and a broken-line function $f_1(x)$ is constructed which is equal to $f(x)$ at the points of division; the proof need not be further recalled.

Combining this fact with Theorem VI, in a manner already sufficiently illustrated, we find at once the generalization:

THEOREM VIII: *If $f(x)$ possesses a continuous kth derivative satisfying the condition that*

$$|f^{(k)}(x_2) - f^{(k)}(x_1)| \leqq \omega(\delta),$$

whenever $|x_2 - x_1| \leqq \delta$ *and* x_1, x_2 *are in the interval* $a \leqq x \leqq b$ *of length* l, *then*

$$f(x) = \mathfrak{P}_n(x) + O\left(\frac{1}{n^k}\omega\left(\frac{l}{n-k}\right)\right)$$

uniformly for all values of x in the interval.

The constant multiplier implied in the O-symbol depends only on k and l; as far as l is concerned, it may be taken proportional to l^k.

The reader is referred to the first two theorems of a recent paper by the author,* with the accompanying discussion, for an indication of the manner in which the above result may be applied in the theory of Legendre's series.

III. Fourier's Series.

We shall be concerned in this section principally with the approximation afforded by trigonometric sums chosen in a particular way, namely, those obtained by breaking off a Fourier's series after the terms of the nth order. In particular we shall investigate the degree of convergence of the series obtained by integrating or differentiating a given Fourier's series a given number of times. If the number of integrations or differentiations performed is even, the problem is very simple, and does not call for methods closely related to those of the earlier parts of the paper.† But if the number is odd, the direct method is no longer sufficient, and Theorem II of the first part is adduced to furnish the needed connection. It is this circumstance which unites the present section with what goes before.‡

* These Transactions, vol. 13 (1912), pp. 305–318.

† This case was treated to some extent in my thesis, pp. 82–86, as far as differentiation is concerned.

‡ Since this paper was written, the author has received, through the kindness of S. Bernstein, a copy of the latter's Mémoire Couronné, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Bruxelles, 1912, in which methods are developed that render possible a more satisfactory discussion of the problems of the present section. The reader of both papers will have little difficulty in seeing how some of the theorems below may be improved in statement. Bernstein is more especially interested, to be sure, in the problem of polynomial approximation, but he provides the instruments for the treatment of the trigonometric case, in particular (p. 20) the theorem:

"S étant une suite trigonométrique quelconque, si son module reste inférieur à L, le module de sa dérivée reste inférieur à $2nL$."

It is a well-known fact, of which use will be made more than once below, that if a function can be represented to a known degree of approximation by a trigonometric sum of specified order, an upper limit for the error of the corresponding partial sum of its Fourier's series can be deduced. With this in mind, it may be seen how the theorems concerning the differentiation of Fourier's series throw light on the degree of approximation by a trigonometric sum of given order attainable in the case of a derivative of a function which can itself be approximately represented by such a sum with known accuracy. A further connection with the first part thus becomes apparent. The approximation to the derivative obtained in this way is not of such an order as to yield a converse of the integration-theorem of Part I, but this is not surprising if we consider the special nature of the trigonometric sums employed.*

The other results of the section are, in the first place, the theorem concerning the degree of convergence of the Fourier's series for a function satisfying certain conditions (Theorem X), with a consequence readily deduced from it (Theorem XI); further, the theorem concerning the convergence of a series associated in a particularly simple way with a given Fourier's series (Theorem XIII); and the concluding theorems,† which give *necessary* conditions for the possibility of certain degrees of approximation.

To begin the detailed investigation, let us suppose that the series

(1) $$(a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

converges uniformly for all values of x to the value‡ $f(x)$. Let us set

$$r_n(x) = \sum_{h=n+1}^{\infty} (a_h \cos hx + b_h \sin hx).$$

Let $\varphi(n)$ be a function of n such that

$$|r_n(x)| \leqq \varphi(n)$$

for all values of x, while $\varphi(n+1) \leqq \varphi(n)$ for each value of n and $\lim_{n=\infty} \varphi(n) = 0$. Such a function will surely exist; for example, $\varphi(n)$ may be defined as the maximum of $|r_h(x)|$ for $h \geqq n$. The function $\varphi(n)$, then, measures the rate of convergence of the series (1), in the sense that it gives an upper limit for the magnitude of the remainder. We shall obtain a function performing the same service with reference to certain other series related to (1).

If q is any positive integer, we have, as an immediate consequence of our hypotheses,

$$|r_{n+q}(x)| \leqq \varphi(n+q) \leqq \varphi(n),$$

* Cf. Bernstein, loc. cit., p. 27.

† These are only partly new; see references given in connection with them below.

‡ It must accordingly be the Fourier's series for $f(x)$.

and hence

$$(2) \qquad \left| \sum_{h=n+1}^{n+q} (a_h \cos hx + b_h \sin hx) \right| = | r_n(x) - r_{n+q}(x) | \overline{\overline{<}} 2\varphi(n) \,;$$

Now let us set

$$\rho_{np}(x) = \sum_{h=n+1}^{n+p} \frac{1}{h^l} (a_h \cos hx + b_h \sin hx),$$

where l is a positive integer. Remembering that (2) holds for all the values $q = 1, \cdots, p$, we see that we have a sum to which Abel's lemma is applicable; we conclude that

$$| \rho_{np}(x) | \overline{\overline{<}} \frac{2\varphi(n)}{(n+1)^l}.$$

As the right-hand member here is independent of p, and approaches zero as n becomes infinite, the series

$$(3) \qquad (a_1 \cos x + b_1 \sin x) + \frac{1}{2^l}(a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

converges uniformly, and if the remainder of this series after the terms in nx is set equal to $\rho_n(x)$, we have

$$| \rho_n(x) | \overline{\overline{<}} \frac{2\varphi(n)}{(n+1)^l}.$$

Now if $l = 2l'$ is an even number, the series (3), multiplied by $(-1)^{l'}$, is the Fourier's series of a function obtained by integrating (1) repeatedly l times.* We may state our conclusion in the following form:

THEOREM IX*a*: *If the remainder after the term of the nth order in the Fourier's series for a function $f(x)$ never exceeds $\varphi(n)$ in absolute value, where $\varphi(n)$ is a function of the nature specified above, then the corresponding remainder in the Fourier's series for a periodic lth integral of $f(x)$, in the case that l is even, does not exceed $2\varphi(n) / (n+1)^l$ in absolute value.*

It is readily seen how this statement has to be modified so as to be applicable to an integral which is not periodic, because of the presence of polynomial terms; this case would surely arise if the Fourier's series for $f(x)$ had a constant term different from zero.

The method just used gives no direct information about the series obtained by integrating (1) an odd number of times. To make progress here, we apply our Theorem II, modified in the manner indicated at the close of its proof. The partial sum of (1) to terms of the nth order is a trigonometric sum without constant term which represents $f(x)$ with an error less than or equal to $\varphi(n)$;

* Cf., e. g., BÔCHER, loc. cit., p. 117. The proof can be made particularly simple here, since all the series concerned are uniformly convergent.

it follows that $\int f(x)\,dx$ can be represented by a trigonometric sum of the nth order with an error not exceeding $3\,\varphi(n)/n$. From this we deduce the further consequence* that the remainder after terms of the nth order, $n \geqq 1$, in the Fourier's series for $\int f(x)\,dx$ does not exceed $[3K\varphi(n)\log n]/n$, where K is a certain absolute constant; if we restrict n to values not less than 5, we may say more specifically that the remainder does not exceed $[6\varphi(n)\log n]/n$. From this point we may apply our earlier reasoning, in fact, precisely the theorem IXa; we find the supplementary result:

THEOREM IXb: *If the number l of Theorem IXa is odd instead of even, the remainder in the Fourier's series for the lth integral does not exceed*

$$\frac{12\varphi(n)\log n}{n(n+1)^{l-1}}$$

in absolute value, provided $n \geqq 5$.

More roughly, the expressions $2\varphi(n)/n^l$ and $[12\varphi(n)\log n]/n^l$ may be assigned as limits of error in the two cases.

To make use of Theorem IXa in another way, let $f(x)$ be a function which fulfils the hypotheses of Theorem III; it is not our purpose, however, to apply the latter theorem to $f(x)$. If the number k which figures in the hypothesis is odd, so that $k-1$ is even, we say at first merely that $f^{(k-1)}(x)$ has the form $\mathfrak{T}_n(x) + \theta(n, x)\cdot 3\lambda/n$, by Theorem I. Hence we infer that $f^{(k-1)}(x)$ is represented by its Fourier sum of the nth order, if we may use this abbreviation for the partial sum of its Fourier's series, with an error never greater than $(6\lambda\log n)/n$ for $n \geqq 5$. By Theorem IXa we pass to $f(x)$ itself, and see that the latter function is represented by its Fourier sum of the nth order with an error not exceeding $(12\lambda\log n)/n^k$. If k is even, and $k-1$ odd, we begin by expressing $f^{(k-2)}(x)$ as $\mathfrak{T}_n(x) + \theta(n, x)\cdot 9\lambda/n^2$, by Theorem III; for the Fourier's series, we get the limits of error $(18\lambda\log n)/n^2$, for $f^{(k-2)}(x)$, and $(36\lambda\log n)/n^k$, for $f(x)$. In other words, we have†

THEOREM X: *If $f(x)$ is a function of period 2π possessing a $(k-1)$th derivative which everywhere satisfies the Lipschitz condition*

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda\,|x_2 - x_1|,$$

then $f(x)$ is everywhere approximately represented by the partial sum of its Fourier's series to terms of the nth order, $n \geqq 5$, with an error not exceeding $(36\lambda\log n)/n^k$. If k is odd, the factor 36 may be replaced by 12.

By applying Theorem III directly to $f(x)$, and passing then to the Fourier's series, we should have found instead of the number 36 a coefficient depending on k, though not on anything else.

* For proof and further references, see the paragraphs immediately preceding Theorems V and X of article A.

† Cf. BERNSTEIN, loc. cit., pp. 92–93, for a broader theorem, without determination of the numerical factor.

A further consequence of the theorem just obtained is perhaps not without interest. It is readily seen that if $t(x)$ is a finite trigonometric sum of the pth order,

$$t(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_p \cos px + b_p \sin px,$$

where a_p and b_p are not both zero, and if two positive constants m and G exist so that

$$|t^{(k)}(x)| \overline{\gtrless} Gm^k$$

for all values of x and for all, or for infinitely many, values of k, then $p \overline{\gtrless} m$. For we have

$$t^{(k)}(x) = (-1)^k [(a_1 \cos x + b_1 \sin x) + \cdots + p^k (a_p \cos px + b_p \sin px)]$$

or

$$t^{(k)}(x) = (-1)^{\frac{k+1}{2}} [(a_1 \sin x - b_1 \cos x) + \cdots + p^k (a_p \sin px - b_p \cos px)]$$

according as k is even or odd. In each of these expressions let us set $x = x_0$, where x_0 is a value such that neither $a_p \cos px_0 + b_p \sin px_0$ nor $a_p \sin px_0 - b_p \cos px_0$ is zero, a value which will surely exist, and divide by m^k. If $p > m$, each quotient becomes infinite* with k, which is contrary to hypothesis.

Now let $f(x)$ be any function of period 2π having derivatives of all orders which satisfy the condition imposed on the derivatives of $t(x)$ above. If k is one of the indices for which $|f^{(k)}(x)| \overline{\gtrless} Gm^k$, it follows from the law of the mean that

$$|f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \overline{\gtrless} Gm^k |x_2 - x_1|,$$

whatever the values of x_1 and x_2 may be. By Theorem X, the function $f(x)$ differs from its Fourier sum of the nth order, $n > 4$, by not more than $(36Gm^k \log n)/n^k$. If we give n a fixed value greater than m, this relation, holding for infinitely many values of k and so for values of k as large as we please, requires that $f(x)$ be identically equal to the Fourier sum. Being a finite trigonometric sum, $f(x)$ can not be of order higher than m, and we have

Theorem XI: *If $f(x)$ is a function of period 2π possessing derivatives of all orders, and if, for infinitely many values of k,*

$$|f^{(k)}(x)| \overline{\gtrless} Gm^k$$

for all values of x, where m and G are constants, then $f(x)$ is a finite trigonometric sum of order not higher than m.

It may be observed that there has been no occasion to exclude the possibility that $0 < m < 1$; in this case, $f(x)$ reduces to a constant.

* This will perhaps be more obvious in case $p > m + 1$ if we divide, not by m^k, but by $(p-1)^k$.

The last paragraphs have been of the nature of a digression. To return to our earlier line of reasoning, suppose again that the series (1) converges uniformly so that $|r_n(x)| \overline{\gtrless} \varphi(n)$ for all values of n and x, but suppose now that φ is a function decreasing so rapidly as n increases that $n^{l+1}\varphi(n)$ decreases monotonically, that is, that $(n+1)^{l+1}\varphi(n+1) \overline{\gtrless} n^{l+1}\varphi(n)$ for every value of n. By l we mean again a positive integer. We set

$$\rho_{np}(x) = \sum_{h=n+1}^{n+p} h^l (a_h \cos hx + b_h \sin hx).$$

We see that this may be rewritten

$$\begin{aligned}\rho_{np}(x) &= \sum_{h=n+1}^{n+p} h^l [r_{h-1}(x) - r_h(x)] \\ &= \sum_{h=n+1}^{n+p} [(h+1)^l - h^l] r_h(x) + (n+1)^l r_n(x) - (n+p+1)^l r_{n+p}(x),\end{aligned}$$

the last expression being derived from the preceding by "partial summation."* In the expression under the sign of summation, we notice that

$$(h+1)^l - h^l = l\xi^{l-1},$$

where ξ is a value between h and $h+1$, and accordingly

$$(h+1)^l - h^l < l(h+1)^{l-1}.$$

Applying this inequality and the fact that $(h+1)/h$ and $h^{l+1}\varphi(h)$ are monotonically decreasing functions, we see that

$$\begin{aligned}\left|\sum_{h=n+1}^{n+p} [(h+1)^l - h^l] r_h(x)\right| &\overline{\gtrless} \sum_{h=n+1}^{n+p} l(h+1)^{l-1}\varphi(h) \\ &= \sum_{h=n+1}^{n+p} l\left(\frac{h+1}{h}\right)^{l-1} h^{l+1}\varphi(h)\cdot\frac{1}{h^2} \overline{\gtrless} l\left(\frac{n+1}{n}\right)^{l-1} n^{l+1}\varphi(n)\sum_{h=n+1}^{n+p}\frac{1}{h^2} \\ &\overline{\gtrless} l\left(\frac{n+1}{n}\right)^{l-1} n^{l+1}\varphi(n)\int_n^\infty \frac{dt}{t^2} = l\left(\frac{n+1}{n}\right)^{l-1} n^l\varphi(n).\end{aligned}$$

For the terms outside the sign of summation in the expression for $\rho_{np}(x)$, we have the inequalities

$$|(n+1)^l r_n(x)| \overline{\gtrless} (n+1)^l\varphi(n) = \left(\frac{n+1}{n}\right)^l n^l\varphi(n),$$

$$\begin{aligned}|(n+p+1)^l r_{n+p}(x)| &\overline{\gtrless} \left(\frac{n+p+1}{n+p}\right)^l (n+p)^l\varphi(n+p) \\ &\overline{\gtrless}' \left(\frac{n+1}{n}\right)^l n^l\varphi(n),\end{aligned}$$

* For a general formulation of this device, cf. LANDAU, *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 84.

the last inequality resulting from the fact that, if $n^{l+1}\varphi(n)$ decreases monotonically, then $n^l\varphi(n)$ has the same property.* Hence

$$|\rho_{np}(x)| \overline{\gtrless} l\left(\frac{n+1}{n}\right)^{l-1} n^l\varphi(n) + 2\left(\frac{n+1}{n}\right)^l n^l\varphi(n) \overline{\gtrless} c_l n^l\varphi(n),$$

where c_l is a constant depending only on l. The last expression is independent of p. Since $n^{l+1}\varphi(n)$ decreases monotonically, it must *a fortiori* remain finite as n becomes infinite, and $n^l\varphi(n)$ must approach zero. It follows that the series

$$\sum_{n=1}^{\infty} n^l(a_n\cos nx + b_n\sin nx) \tag{4}$$

converges uniformly. Also, of course, $\rho_{np}(x)$ approaches uniformly the limit

$$\rho_n(x) = \sum_{h=n+1}^{\infty} h^l(a_h\cos hx + b_h\sin hx)$$

as p becomes infinite. Passing to the limit in the last inequality for $|\rho_{np}|$, we find that

$$|\rho_n(x)| \overline{\gtrless} c_l n^l\varphi(n).$$

If l is even, the series (4) is, except perhaps as to sign, the Fourier's series of a continuous lth derivative of $f(x)$, as we shall again call the sum of (1). For a series of this form, when uniformly convergent, is always the Fourier's series of the function which it represents, and we can get (1) from (4) by repeated term-by-term integration. We have proved†

THEOREM XIIa: *If the absolute value of the remainder after terms of the nth order in the Fourier's series for a function‡ $f(x)$ never exceeds $\varphi(n)$, where φ is such that $n^{l+1}\varphi(n)$ decreases monotonically, then the corresponding remainder in the Fourier's series for the lth derivative of $f(x)$, which will surely exist, does not exceed $c_l n^l\varphi(n)$ in absolute value in the case that l is even, where c_l is a constant depending only on l.*

If, in the hypothesis, we had assumed somewhat less, namely, only that $n^l\varphi(n)\log^2 n$ decreases monotonically, we should still have been able to deduce the uniform convergence of (4) with a remainder, that does not exceed $n^l\varphi(n)\log n$ multiplied by a constant depending on l. It is clearly possible to go further in the same direction.

Let us see what happens if l in the hypothesis of Theorem XIIa is odd. The integral $\int f(x)\,dx$, where f has been modified, if necessary, by the subtrac-

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* For $(n+1)^l\varphi(n+1) = \frac{1}{n+1}(n+1)^{l+1}\varphi(n+1) \leqq \frac{1}{n}n^{l+1}\varphi(n) = n^l\varphi(n)$.

† In connection with Theorems XIIa and XIIb, cf. BERNSTEIN, loc. cit., pp. 22–28.

‡ The fact that the series (1) with which we have been working has no constant term is clearly immaterial.

tion of a constant so that the integral shall be periodic, can be represented by a trigonometric sum of the nth order at most with an error never greater than $3\varphi(n)/n$, and is represented by its Fourier sum of the nth order with an error not exceeding* $[3K\varphi(n)\log n]/n$ for $n \overline{>} 2$. We see that we shall want to modify the hypothesis still further to the extent of supposing that $n^{l+1}\varphi(n)\log n$ decreases monotonically; then we can apply Theorem XII*a*, with l replaced by $l+1$, to $\int f(x)\,dx$, and obtain

THEOREM XII*b*: *If the number l of Theorem XIIa is odd instead of even, and if φ is such that $n^{l+1}\varphi(n)\log n$ decreases monotonically, then the remainder in the Fourier's series for $f^{(l)}(x)$ does not exceed $C_l n^l \varphi(n)\log n$ in absolute value, when $n \overline{>} 2$; here C_l is a constant depending only on l.*

In this connection, we observe that from a knowledge of the degree of convergence of the series

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{5}$$

information can be derived as to the convergence of a certain series which is neither its integral nor its derivative, namely the series

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx). \tag{6}$$

Particularly interesting cases are those in which all the a's or all the b's are zero. If the first series merely converges uniformly, the second may diverge for some values of the argument, as may be shown by examples.† We shall prove, however, that the following is true:‡

THEOREM XIII: *If the absolute value of the remainder after terms of the nth order in the series* (5) *is never greater than $\varphi(n)$, where φ is such a function that*

$$\sum_{n=1}^{\infty}\frac{\varphi(n)\log n}{n}$$

converges and that $\lim_{n=\infty}\varphi(n)\log n = 0$, then the series (6) *converges uniformly.*

These conditions would be satisfied, for example, if $\varphi(n)$ were a constant multiple of $1/(\log^{2+\delta} n)$, δ being any positive quantity.

Let the sum of (5) be denoted by $f(x)$. The Fourier's series

$$\int f(x)\,dx = \sum_{n=1}^{\infty}\frac{1}{n}(a_n \sin nx - b_n \cos nx),$$

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* Cf. the proof of Theorem IX*b*.

† Cf., e. g., FEJÉR, *Über gewisse Potenzreihen an der Konvergenzgrenze*, Sitzungsberichte der Bayerischen Akademie (1910), 3. Abhandlung, pp. 10–11.

‡ For related but different theorems, cf. BERNSTEIN, loc. cit., pp. 95–96.

the constant of integration being suitably chosen, converges uniformly; and, as we assure ourselves through the medium of Theorem II, its remainder

$$\rho_n(x) = \sum_{h=n+1}^{\infty} \frac{1}{h}(a_h \sin hx - b_h \cos hx)$$

satisfies the inequality

$$|\rho_n(x)| \leqq \frac{3K\varphi(n)\log n}{n}$$

for $n \geqq 2$. Let

$$R_{np}(x) = \sum_{h=n+1}^{n+p} (a_h \sin hx - b_n \cos hx).$$

Then

$$\begin{aligned} R_{np}(x) &= \sum_{h=n+1}^{n+p} h[\rho_{h-1}(x) - \rho_h(x)] \\ &= \sum_{h=n+1}^{n+p} [(h+1)-h]\rho_h(x) + (n+1)\rho_n(x) - (n+p+1)\rho_{n+p}(x) \\ &= \sum_{h=n+1}^{n+p} \rho_h(x) + (n+1)\rho_n(x) - (n+p+1)\rho_{n+p}(x); \end{aligned}$$

$$|R_{np}(x)| \leqq \sum_{h=n+1}^{\infty} \frac{3K\varphi(h)\log h}{h} + 3K\frac{n+1}{n}\varphi(n)\log n + 3K\frac{n+p+1}{n+p}\varphi(n+p)\log(n+p).$$

Let ϵ be any positive number, and let N be so large that for $n > N$

$$3K\frac{n+1}{n}\varphi(n)\log n < \frac{\epsilon}{3},$$

and

$$\sum_{h=n+1}^{\infty} \frac{3K\varphi(h)\log h}{h} < \frac{\epsilon}{3}.$$

This is surely possible, with the hypotheses that we have adopted. Then we have also

$$3K\frac{n+p+1}{n+p}\varphi(n+p)\log(n+p) < \frac{\epsilon}{3},$$

since $n + p > n$, and this inequality, combined with the preceding ones, shows that

$$|R_{np}(x)| < \epsilon,$$

if $n > N$. It follows that the series (6) does converge uniformly, as was asserted.

With our earlier results at our command, we are in a position to supplement a theorem of my thesis* concerning the existence of derivatives as a necessary

* *Thesis*, Satz XVIII*a*, concerning derivatives of even order.

condition for the possibility of certain degrees of approximation by trigonometric sums, and to prove by our methods a theorem of S. BERNSTEIN* relating to the corresponding problem in polynomial approximation.

We suppose that a function $f(x)$ of period 2π can be represented in the form $\mathfrak{T}_n(x) + O(\varphi(n))$ for all positive integral values of n, and that $n^l \varphi(n) \log^3 n$ is a monotonically decreasing function; this condition is satisfied, for example, if $\varphi(n) = C / (n^l \log^4 n)$, where C is an arbitrary constant.

Since the remainder in the Fourier's series for $f(x)$ is uniformly $O(\varphi(n) \log n)$, it is a consequence of Theorem XIIa, refined in the manner suggested in the remark following it, that $f(x)$ has everywhere a continuous lth derivative when l is even.

If l is odd, the same conclusion can be drawn without alteration in the hypothesis. The integral† $\int f(x)\,dx$ has the form $\mathfrak{T}_n(x) + O(\varphi(n)/n)$, and is represented by its Fourier sum of the nth order with an error which is $O[\varphi(n)(\log n)/n]$. Hence, by the refined Theorem XIIa, the integral has a continuous derivative of order $l+1$, and $f(x)$ has a continuous derivative of order l. To summarize:

THEOREM XIV: *If $f(x)$ can be approximately represented by a trigonometric sum of the nth order at most for all positive integral values of n with an error not exceeding $\varphi(n)$, where φ is such that $n^l \varphi(n) \log^3 n$ decreases monotonically, then $f(x)$ has everywhere a continuous lth derivative.*

It will readily be seen how the hypothesis may be replaced by slightly more general ones, for instance, by the assumption that $n^l \varphi(n) (\log\log n)^2 \log^2 n$ decreases monotonically.

For the polynomial case, suppose that $f(x)$ has the form $\mathfrak{P}_n(x) + O(\varphi(n))$, uniformly throughout some interval $a \overline{\gtrless} x \overline{\gtrless} b$. Without loss of generality, we may assume that the interval is that from -1 to $+1$. We assume again that $n^l \varphi(n) \log^3 n$ decreases monotonically. As $f(x)$ is represented by a polynomial in x, so‡ $f(\cos x)$ is represented by a polynomial in $\cos x$; and a polynomial of the nth degree in $\cos x$ is a trigonometric sum of the nth order in x. Hence $f(\cos x)$ has a continuous lth derivative with respect to x, for all values of x. It follows that $f(\cos x)$ has a continuous derivative with respect to $\cos x$ at all points except possibly those at which $dx/d(\cos x)$ becomes infinite; that is, at all points where $\cos x \neq \pm 1$. Hence we have the theorem:

* S. BERNSTEIN, Comptes Rendus, vol. 152 (1911), pp. 502–504; cf. *Thesis*, Satz XVIII. The theorem of BERNSTEIN is somewhat more general than that which we shall obtain; the theorem of the thesis is essentially less so, relating only to derivatives of even order.

A detailed discussion, including the trigonometric case, is given by BERNSTEIN in Chapter II, pp. 22–36, of the essay referred to in previous footnotes.

† Here again it may be necessary to begin by subtracting a constant from $f(x)$, an operation which clearly does not impair the correctness of the ultimate result.

‡ Cf. *Thesis*, p. 85.

THEOREM XV: *If $f(x)$ can be approximately represented throughout a closed interval by a polynomial of the nth degree at most for all positive integral values of n with a maximum error not exceeding $\varphi(n)$, where φ is such that $n^l \varphi(n) \log^3 n$ decreases monotonically, then $f(x)$ has a continuous lth derivative throughout the interval, with the possible exception of the end-points.*

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